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Renormalisation group calculations on a mixed-spin system in two dimensions

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Abstract. The real space renormalisation group of Niemeijer and van Leeuwen is applied to a mixed-spin Ising model on a simple quadratic lattice. The motivation is the investigation of critical phenomena in Ising models with less than the usual translational symmetry. The models in question are relevant to the study of ferrimagnetism. Two calculations, characterised by different block constructions, are performed and compared. Exponent values are found to be in good agreement with those suggested by the universality hypothesis. The utility of the renormalisation group for dealing with ferrimagnetism is demonstrated, but the high degree of labour involved in such an exercise is indicated.

1. Introduction

Since the introduction, by Wilson (1971a, b), of the renormalisation group (RG) method for the analysis of critical phenomena, attention has been largely focused on relatively simple model systems. In particular, there has been extensive study of systems designed to simulate the critical properties of ferro- and antiferromagnets. In this respect, analyses of the Ising and Heisenberg model have been predominant. After Wilson's work on systems with continuous spin variables (in which the spatial dimension $d = 4$ plays an important part, and application to two-dimensional systems is highly questionable), there followed the development of realisations of the RG specifically designed to treat systems of discrete spins and ideally suited to two dimensions (Barber 1977). All of these methods have so far dealt with systems in which all lattice sites are equivalent in the sense that only a single kind of spin is present. This restriction has limited the analyses to ferro- and antiferromagnetism, and it appears desirable to extend the scope of investigation to include the case of ferrimagnetism (Néel 1948). The aim of this paper is to go some way towards achieving this objective through a RG analysis of a particularly simple model capable of uni-axial ferrimagnetism.

Of the discrete spin treatments mentioned above, the cumulant expansion of Niemeijer and van Leeuwen (1973, 1974) is employed in this paper. This particular method was first applied, by Niemeijer and van Leeuwen (1973, 1974), to the case of a two-dimensional spin-$\frac{1}{2}$ Ising model on a triangular lattice, and this work was later extended by Sudbo and Hemmer (1976), and Hemmer and Velarde (1976), who provided some insight into the asymptotic properties of the perturbation expansion on which the method is based. Further insight into the nature of the method was supplied

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through analysis, by Nauenberg and Nienhuis (1974), of the spin-$\frac{1}{2}$ model on a square lattice, and similar investigations were undertaken by Hsu *et al* (1975). Fields and Fogel (1975) carried out an analysis of the spin-1 Ising model on both a square lattice (two dimensions) and a cubic lattice (three dimensions), and thereby provided some indication of the computational difficulties involved in employing the method to treat three-dimensional problems.

All of these calculations, together with applications of the Niemeijer–van Leeuwen method to simple models with spin dimensionality $n > 1$ (e.g., the classical XY model on the triangular lattice, Lublin (1975)), have indicated the manner in which the universality of critical exponents arises naturally from the RG formalism. The system considered in this paper is a two-dimensional Ising model consisting of both spin-$\frac{1}{2}$ and spin-1 objects, which one expects to belong to the same universality class (Wilson 1971a, b) as the above (more simple) Ising models. Such a property is borne out by the calculations of this paper, which provide evidence supporting the expected similarities between critical phenomena associated with ferro- and antiferromagnetism on the one hand and ferrimagnetism on the other.

It is important to realise that the Niemeijer–van Leeuwen method is particularly suited to the treatment of the system considered here, since the spin magnitudes of the two species (spin 1 and spin $\frac{1}{2}$) are preserved under the RG transformation, which is just what is wanted. (This contrasts with the ϵ expansion of Wilson and Kogut (1974), where even spin discreteness is lost.) However, we demonstrate in this paper the high degree of labour required to implement the method in the case of systems more complicated than those studied by previous workers, and we are thereby provided with an indication of the practical limitations of this method.

2. Niemeijer–van Leeuwen method on mixed-spin system

The mixed-spin system is illustrated in figure 1, and consists of a simple quadratic lattice of spin-1 and spin-$\frac{1}{2}$ Ising objects, where each spin 1 has only spin $\frac{1}{2}$'s as nearest neighbours, and vice versa. This model is designed so as to be capable of a particular form of uni-axial ferrimagnetism in which there exist two interpenetrating sublattices. The spin-1 and spin-$\frac{1}{2}$ objects are indicated by a circle (○) and cross (×), respectively,

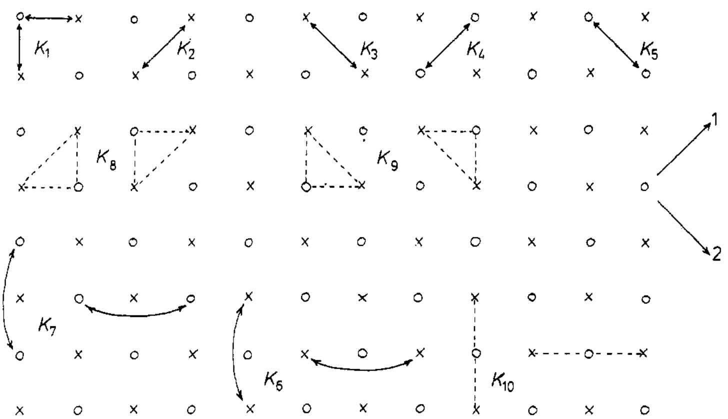
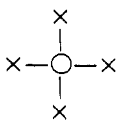
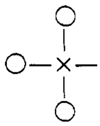


Figure 1. Mixed-spin system and interactions generated in second order.

and the corresponding spin magnitudes are denoted by s and σ, so that $s = -1, 0, 1$ and $\sigma = -\frac{1}{2}, \frac{1}{2}$ are the allowed values. Considering only nearest-neighbour interactions, the reduced Hamiltonian of the system takes the form

$$\mathscr{H}\{\sigma, s\} = K \sum_{\langle ij \rangle} \sigma_i s_j + H\Big(\sum_i \sigma_i + \sum_j s_j\Big) \tag{1}$$

where K and H denote, respectively, the reduced nearest-neighbour coupling constant and the reduced externally applied magnetic field.

We perform in this paper two independent Niemeijer–van Leeuwen RG analyses of the system characterised by two distinct block constructions shown in figures 2 and 3, respectively. Considering the second type of construction for the moment, the spin-1 and spin-$\frac{1}{2}$ blocks are constructed as  and , respectively, and the conventional prescription, given by Niemeijer and van Leeuwen (1973, 1974), for the

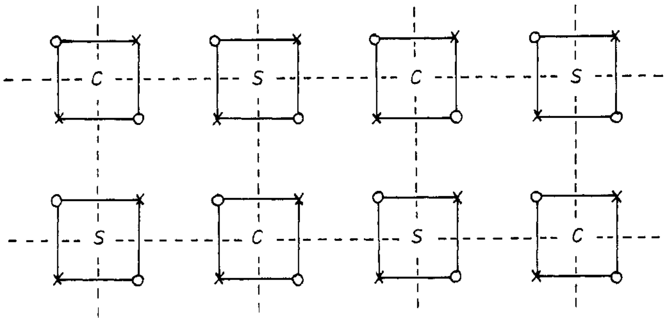


Figure 2. Construction of four-spin blocks.

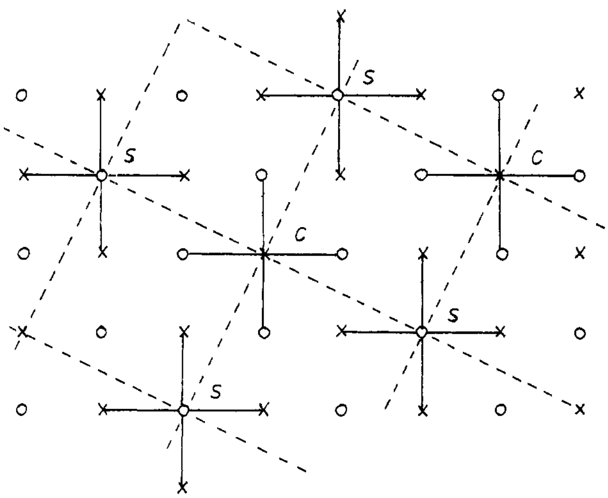


Figure 3. Construction of five-spin blocks.

definition of block spin can readily be applied. Then the spins C_I and S_J of spin-$\frac{1}{2}$ and spin-1 blocks are defined as

$$C_I = \tfrac{1}{2}\,\mathrm{sgn}\left(\sigma^I + \sum_{j=1}^{4} s_j^I\right) \tag{2}$$

and

$$S_J = \mathrm{sgn}\left(s^J + \sum_{i=1}^{4} \sigma_i^J\right) \tag{3}$$

where I and J index the block positions. These definitions lead to the allowed (and required) values $C_I = -\frac{1}{2}, \frac{1}{2}$ and $S_J = -1, 0, 1$, as can readily be seen from a consideration of the $3^4 \times 2$ and $2^4 \times 3$ internal spin configurations appropriate to I and J respectively (it is noted here that we take sgn $(0) = 0$ in (3)).

The RG transformation for a particular block construction is given by

$$\exp\left(\mathcal{H}'(\{C, S\})\right) = \sum_{\{\sigma, s\}} P[C, S; \sigma, s] \exp\left(\mathcal{H}(\{\sigma, s\})\right) \tag{4}$$

where the summation is over all spin states and $P[C, S; \sigma, s]$ is a product over all spin-$\frac{1}{2}$ and spin-1 blocks. For the block construction embodied by (2) and (3), this is given by

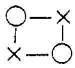
$$P[C, S; \sigma, s] = \prod_{I,J} p^{(1/2)}(C_I; \sigma^I, \{s_j^I, j = 1, 4\}) p^{(1)}(S_J; s^J, \{\sigma_i^J, i = 1, 4\}) \tag{5}$$

where the quantities $p^{(1/2)}$ and $p^{(1)}$ reflect (2) and (3) through the relations

$$p^{(1/2)}(C_I; \sigma^I, \{s_j^I, j = 1, 4\}) = \delta\left[C_I, \tfrac{1}{2}\,\mathrm{sgn}\left(\sigma^I + \sum_{j=1}^{4} s_j^I\right)\right] \tag{6}$$

$$p^{(1)}(S_J; s^J, \{\sigma_i^J, i = 1, 4\}) = \delta\left[S_J, \mathrm{sgn}\left(s^J + \sum_{i=1}^{4} \sigma_i^J\right)\right] \tag{7}$$

and we have employed the Kronecker delta.

Consider now the first type of block construction. It is seen from figure 2 that the construction  is used for both spin-1 and spin-$\frac{1}{2}$ blocks. However, this does not lead to any natural distinction between the blocks, and we find it necessary to impose this formally. This is achieved by defining a spin-1 block in the usual way, i.e. through the relation

$$S_J = \mathrm{sgn}\left(\sum_{i=1}^{2} \sigma_i^J + \sum_{j=1}^{2} s_j^J\right), \tag{8}$$

and defining a spin-$\frac{1}{2}$ block in such a way that those internal spin configurations $\{\sigma^I, s^I\}$ which satisfy $\sum_{i=1}^{2} \sigma_i^I + \sum_{j=1}^{2} s_j^J = 0$ are formally considered to contribute to $C_I = \pm\frac{1}{2}$. The idea of associating such configurations with a non-zero net block spin was introduced by Nauenberg and Nienhuis (1974). Our prescription differs from theirs, however, in that we associate all such configurations (10 out of a total of $2^2 \times 3^2$ for the given construction) with both $C_I = -\frac{1}{2}$ and $C_I = +\frac{1}{2}$, and introduce a multiplying factor of $\frac{1}{2}$ in order to avoid counting these twice. This definition—which seems superior on grounds of symmetry—together with (8) allows the formal construction of the mixed-spin block system of figure 2. Equations (4) and (5) again hold, with the latter modified in such a way that $p^{(1/2)}$ and $p^{(1)}$ are now given by

$$p^{(1/2)}(C_I; s_j^I; \sigma_i^I, i, j=1,2)$$
$$= \delta\left[C_I, \tfrac{1}{2}\operatorname{sgn}\left(\sum_{i=1}^{2}\sigma_i^I + \sum_{j=1}^{2} s_j^I\right)\right] + \tfrac{1}{2}\delta\left[0, \operatorname{sgn}\left(\sum_{i=1}^{2}\sigma_i^I + \sum_{j=1}^{2} s_j^I\right)\right] \tag{9}$$

$$p^{(1)}(S_J; s_j^J, \sigma_i^J, i, j=1,2) = \delta\left[S_J, \operatorname{sgn}\left(\sum_{i=1}^{2}\sigma_i^J + \sum_{j=1}^{2} s_j^J\right)\right]. \tag{10}$$

Following Niemeijer and van Leeuwen (1973, 1974), a calculation proceeds by firstly considering the Hamiltonian (1) with the field term absent, and dividing this into an intrablock part $\mathcal{H}_0$ and an interblock part V, and writing

$$\exp(\mathcal{H}'(\{C, S\})) = \langle\exp(V)\rangle_0 \sum_{\{\sigma,s\}} \exp(\mathcal{H}_0(\{C, S; \sigma, s\})) \tag{11}$$

$$\langle\exp(V)\rangle_0 = \exp[\langle V\rangle_0 + \tfrac{1}{2}(\langle V^2\rangle_0 - \langle V\rangle_0^2) + \ldots] \tag{12}$$

where the sum in (11) is over all spin states compatible with the configuration $\{C, S\}$ of block spins, and $\mathcal{H}'(\{C, S\})$ is the renormalised Hamiltonian of (4). This is expressed in terms of the block spins only, since spin configurations internal to the blocks are summed out in (4) and (11). The brackets are averages analogous to those defined by Niemeijer and van Leeuwen (1973, 1974), and are given by

$$\langle\ldots\rangle_0 = \frac{\sum_{\{\sigma,s\}}(\ldots)\exp(\mathcal{H}_0(\{C, S; \sigma, s\}))}{\sum_{\{\sigma,s\}}\exp(\mathcal{H}_0(\{C, S; \sigma, s\}))}, \tag{13}$$

where the summations are as at (11).

The analyses of this paper are taken up to the second-order term in V of (12), and the Hamiltonian required to generate the corresponding recursion relations for the coupling constants (Niemeijer and van Leeuwen 1973, 1974) contains more terms than in the zero-field part of (1). This is demonstrated explicitly in §§ 3 and 4, where the first and second types, respectively, of block construction are employed.

The recursion relations for the case of non-zero external magnetic field H are found, following Niemeijer and van Leeuwen (1973, 1974), from consideration of the expansion

$$\delta\mathcal{H}'(\{C, S\}) = \delta H_\beta \sum_{\Omega\in\beta} [\langle U_\Omega\rangle + \langle(U_\Omega - \langle U_\Omega\rangle)V\rangle + \ldots] \tag{14}$$

where the quantities U_Ω denote odd-spin interactions of types indexed by β, with H_β the corresponding coupling constants. The analyses are taken up to the first-order term in V of (14) and it is found necessary (for both block constructions) to consider the application of distinct magnetic fields on the two sublattices of the mixed-spin system, together with a three-spin interaction term.

3. Calculation employing blocks of four spins

For the block construction of figure 2, the Hamiltonian required to generate the full second-order approximation to the RG transformation (11) is given by

$$\begin{aligned}\mathcal{H}(\{\sigma, s\}) = K_1 \sum_{\langle ij\rangle} \sigma_i s_j + K_2 \sum_{i2j}{}^{(1)} \sigma_i\sigma_j + K_3 \sum_{i2j}{}^{(2)} \sigma_i\sigma_j + K_4 \sum_{i2j}{}^{(1)} s_i s_j \\ + K_5 \sum_{i2j}{}^{(2)} s_i s_j + K_6 \sum_{i3j} \sigma_i\sigma_j + K_7 \sum_{i3j} s_i s_j + K_8 \sum_{\substack{\langle ij\rangle \\ \langle jk\rangle \\ i2k}}{}^{(1)} \sigma_i s_j s_j \sigma_k \\ + K_9 \sum_{\substack{\langle ij\rangle \\ \langle jk\rangle \\ i2k}}{}^{(2)} \sigma_i s_j s_j \sigma_k + K_{10} \sum_{\substack{\langle ij\rangle \\ \langle jk\rangle \\ i3k}} \sigma_i s_j s_j \sigma_k. \end{aligned} \tag{15}$$

The notation of (15) is explained with the aid of figure 1. The first sum of (15) is over all nearest-neighbour pairs; the second and third sums are over all second-neighbour pairs of spin $\frac{1}{2}$'s aligned, respectively, along the directions 1 and 2 of figure 1, and the fourth and fifth sums are defined similarly with respect to spin 1's; the sixth and seventh sums are over all third-neighbour pairs of spin $\frac{1}{2}$'s and spin 1's, respectively; the eighth and ninth sums are taken over all triplets $\{\sigma_i s_j \sigma_k\}$ such that the pairs ij and jk are nearest neighbour and ik is second neighbour along the directions 1 and 2, respectively; the tenth sum is again over triplets, and ik is third neighbour. It is seen that the coupling constants K_8, K_9 and K_{10} are appropriate to four-spin interactions. It is noted here that the interactions appearing in figure 1 are labelled by the corresponding coupling constants.

We now give the full set of second-order recursion relations. These are

$$\begin{aligned}
K_1' &= 2\psi_1\phi_2 K_1 + \psi_1\phi_1(K_3 + 2K_6) + \psi_2\phi_2(K_4 + 2K_7) + 2\psi_7\phi_1(K_9 + K_{10}) \\
K_2' &= \psi_1^2 K_2 + 2\psi_1^2\phi_3^{(0)} K_8 + (\psi_1^2\phi_3^{(0)} + \psi_2^2\phi_5^{(0)} + 2\psi_1\psi_2\phi_4^{(0)})K_1^2 \\
K_3' &= \psi_2^2 K_5 + (\psi_1^2\phi_6^{(0)} + \tfrac{1}{4}\psi_2^2 + 2\psi_1\psi_2\phi_4^{(0)})K_1^2 \\
K_4' &= \phi_1^2 K_2 + 2\phi_1^2\psi_3 K_8 + (\phi_1^2\psi_3 + \phi_2^2\psi_5 + 2\phi_1\phi_2\psi_4 - \psi_1^2\phi_2^2)K_1^2 \\
K_5' &= \phi_2^2 K_5 + (\phi_1^2\psi_6 + \tfrac{1}{4}\phi_2^2 + 2\phi_1\phi_2\psi_4 - \psi_1^2\phi_2^2)K_1^2 \\
K_6' &= \tfrac{1}{2}(\psi_1^2\phi_6^{(0)} + \psi_2^2\phi_5^{(0)} + 2\psi_1\psi_2\phi_4^{(0)})K_1^2 \\
K_7' &= \tfrac{1}{2}(\phi_1^2\psi_6 + \phi_2^2\psi_5 + 2\phi_1\phi_2\psi_4 - \psi_1^2\phi_2^2)K_1^2 \\
K_8' &= \psi_1^2(\phi_3^{(1)} - \phi_3^{(0)})K_8 + \tfrac{1}{2}[\psi_1^2(\phi_3^{(1)} - \phi_3^{(0)}) + 2\psi_1\psi_2(\phi_4^{(1)} - \phi_4^{(0)}) \\
&\quad + \psi_2^2(\phi_5^{(1)} - \phi_5^{(0)}) - 4\psi_1^2\phi_2^2]K_1^2 \\
K_9' &= \tfrac{1}{2}[2\psi_1\psi_2(\phi_4^{(1)} - \phi_4^{(0)}) + \psi_1^2(\phi_6^{(1)} - \phi_6^{(0)}) - 4\psi_1^2\phi_2^2]K_1^2 \\
K_{10}' &= \tfrac{1}{2}[2\psi_1\psi_2(\phi_4^{(1)} - \phi_4^{(0)}) + \psi_2^2(\phi_5^{(1)} - \phi_5^{(0)}) + \psi_1^2(\phi_6^{(1)} - \phi_6^{(0)}) - 4\psi_1^2\phi_2^2]K_1^2.
\end{aligned} \tag{16}$$

Here, the quantities ψ_α, ϕ_α, $\phi_\alpha^{(1)}$, $\phi_\alpha^{(0)}$ are obtained from the evaluation of $\langle\ldots\rangle_0$, where the brackets contain spins or products thereof. For example, $\langle s_\alpha^{I2}\rangle_0 = \psi_3$ for I a spin-$\frac{1}{2}$ block, and $\langle s_\alpha^{J2}\rangle_0 = (\phi_3^{(1)} - \phi_3^{(0)})S_J^2 + \phi_3^{(0)}$ for J a spin-1 block, where ψ_3, $\phi_3^{(1)}$ and $\phi_3^{(0)}$ depend on the coupling parameters K_1, K_2, K_5 and K_8. The derivation of (16) together with these quantities can be found in Schofield (1980).

In dealing with the presence of a magnetic field H, the 'field' part of the Hamiltonian required to take consideration of (14) up to and including the first-order term is given by

$$\mathcal{H}^{(\text{field})}(\{\sigma, s\}) = H_1 \sum_i \sigma_i + H_2 \sum_j s_j + H_3 \sum_{\langle ij\rangle} \sigma_i s_j s_j \tag{17}$$

where the first two expressions deal with distinct (reduced) magnetic fields H_1 and H_2 applied to the respective sublattices, and the third expression is a sum, over all nearest-neighbour pairs, of three-spin interactions with coupling parameter H_3. The first-order recursion relations are given, in matrix form, by

$$\boldsymbol{\delta H}' = \boldsymbol{A} \times \boldsymbol{\delta H} \tag{18}$$

where $\boldsymbol{\delta H} = (\delta H_1, \delta H_2, \delta H_3)^{\mathrm{T}}$ (and similarly for $\boldsymbol{\delta H}'$) and $\boldsymbol{A}$ is a (3×3) matrix whose elements are expressions, involving the quantities ψ and ϕ above, evaluated at the zero-field fixed point $(K_1^*, \ldots, K_{10}^*)$ obtained from (16). These expressions are complicated and are not presented here; they can be found in Schofield (1980).

First-order zero-field results are obtained from (16) on substituting $K_i = 0$, $i = 2, \ldots, 10$. A non-trivial fixed point $K^* = K_c$ is readily found, and is listed in table 1, together with the eigenvalue $\lambda_T = (\partial K'/\partial K)_{K^*}$. (In this table superscripts refer to the order of the calculation.)

Table 1. Results of four-spin block calculation.

$\lambda_T^{(1)}$	$\lambda_T^{(2)}$	$\lambda_H^{(0)}$	$\lambda_H^{(1)}$	$K_c^{(1)}$	$K_c^{(2)}$
1·960	2·023	2·677	3·600	1·289	1·186
$\alpha^{(1)}$	$\alpha^{(2)}$	β	γ	$\delta^{(0)}$	$\delta^{(1)}$
−0·060	0·033	0·150	1·669	2·453	12·153

The first-order recursion relation for the nearest-neighbour coupling parameter K is unchanged under the transformation $K \to -K$ (Schofield 1980). It follows from this that $-K_c^{(1)}$ is also a fixed point with corresponding eigenvalue identical to that already obtained from $K_c^{(1)}$. It is now seen that the eigenvalue λ_T relevant to the case of the nearest-neighbour anti-aligning ferrimagnet (characterised by $K < 0$) can be obtained from a first-order study of the corresponding ferromagnet ($K > 0$). This finding is consistent with an exact result, for the nearest-neighbour model, given by Schofield (1980). For this model one can use a sublattice spin reversal (e.g. $s_j \to -s_j$, $\forall j$) to show that the properties of the ferromagnet in uniform field (H, H) are identical to those of the ferrimagnet in staggered field $(H, -H)$ where the notation indicates the fields acting on the individual sublattices.

An analogous result to this can readily be shown to hold when the even-body interactions of figure 1 are present. In this case ferro- and ferrimagnetism can be characterised, respectively, by $(K_1 > 0, K_2, \ldots, K_{10})$ and $(K_1 < 0, K_2, \ldots, K_{10})$. The second-order recursion relations (16) are not, however, unchanged under the transformation $(K_1, K_2, \ldots, K_{10}) \to (-K_1, K_2, \ldots, K_{10})$ and consequently do not possess the symmetry embodied in the exact result (in contrast to the first-order case above). Nevertheless, the exact result indicates that consideration of the appropriate fixed point $(K_1^* > 0, K_2^*, \ldots, K_{10}^*)$ will yield critical exponents whose values are the same (within numerical approximation) as those obtained from a corresponding point with negative nearest-neighbour coupling parameter. With these comments in mind, we look for a fixed point $(K_1^* > 0, K_2^*, \ldots, K_{10}^*)$ in this paper. In the subsequent analyses, we regard the derived exponents as being appropriate either to ferromagnetism in a uniform field or to ferrimagnetism in a staggered field.

A fixed point of (16) of the above type is readily found by numerical means, and is given by

$$(K_1^*, \ldots, K_{10}^*) = (1{\cdot}0443, -0{\cdot}0063, 0{\cdot}3000, 0{\cdot}0768, 0{\cdot}1471, -0{\cdot}2736, -0{\cdot}0021, 0{\cdot}0860, 0{\cdot}0356, 0{\cdot}2511). \quad (19)$$

Linearisation of the RG transformation about the fixed point yields a (10×10) matrix of coefficients $(\partial K_i/\partial K_j)_{K^*}$, the eigenvalues of which are found to be

$$\begin{array}{lll} \lambda_T = 2{\cdot}0231 & \lambda_2 = 0{\cdot}2428 & \lambda_3 = 0{\cdot}0675 + 0{\cdot}0643\mathrm{i} \\ \lambda_4 = \lambda_3^* & \lambda_5 = 0{\cdot}5696 & \lambda_6 = \lambda_7 = \lambda_8 = \lambda_9 = \lambda_{10} = 0 \end{array} \quad (20)$$

where * denotes complex conjugation.

The quantity $K_c = J/kT_c$, where T_c denotes the critical temperature, is determined from (19) and the left eigenvector corresponding to the relevant eigenvalue λ_T using the linear extrapolation procedure of Niemeijer and van Leeuwen (1973, 1974), and is listed in table 1.

For the magnetic field analysis, only the first two terms of (17) are required in a zeroth-order approximation to (14), and (18) is reduced to consideration of a (2×2) matrix, for which we find a single non-zero eigenvalue λ_H. This is given in table 1. The (3×3) matrix $\boldsymbol{A}$ associated with the first-order analysis is found to have one relevant eigenvalue λ_H (also listed in table 1) and two irrelevant eigenvalues, $\lambda_2 = 0{\cdot}0188 + 0{\cdot}1367\mathrm{i}$ and $\lambda_3 = \lambda_2^*$.

The best estimates of λ_T and λ_H, obtained from the zero-field second-order and non-zero field first-order analyses, respectively, are employed to yield the exponents α, β, γ and δ. Only two are independent; the others are given by the exponent scaling laws which arise naturally from the RG formalism (Wilson 1971a, b). The results are presented in table 1, together with those for α and δ obtained from the lower-order analyses. It is seen that the best estimates are in good agreement with the values $\alpha = 0$, $\beta = 0{\cdot}125$, $\gamma = 1{\cdot}75$, $\delta = 15$ suggested by the universality hypothesis.

4. Calculation employing blocks of five spins

For the block construction of figure 3, the Hamiltonian required to generate the second-order recursion relations is analogous to (15), and is obtained from the latter through the removal of the distinction between the directions 1 and 2. We now have $K_3 = K_2$, $K_5 = K_4$ and $K_9 = K_8$, and it is seen that $\mathscr{H}$ is described by the seven coupling constants $(K_1, K_2, K_4, K_6, K_7, K_8, K_{10})$. On relabelling these $(K_1, \ldots, K_7)$ the full set of second-order recursion relations are given by

$$\begin{aligned} K_1' &= 3\psi_2\phi_1K_1 + \psi_2\phi_2(K_2+K_4) + \psi_1\phi_1(K_3+K_5) + \psi_7\phi_1(2K_6+K_7) \\ K_2' &= \phi_1^2(K_3+2K_5) + 2\phi_1^2\psi_3(K_6+K_7) + \phi_1^2(5\psi_6 + 2\psi_{61} + 2\psi_3 - \tfrac{9}{4}\psi_2^2)K_1^2 \\ K_3' &= \psi_2^2(K_2+2K_4) + \psi_2^2(5\phi_5^{(0)} + 2\phi_6^{(0)} + \tfrac{1}{2})K_1^2 \\ K_4' &= \tfrac{1}{2}\phi_1^2(4\psi_6 + 5\psi_{61} - \tfrac{9}{4}\psi_2^2)K_1^2 \\ K_5' &= \tfrac{1}{2}\psi_2^2(4\phi_5^{(0)} + 5\phi_6^{(0)})K_1^2 \\ K_6' &= \tfrac{1}{2}\psi_2^2[5(\phi_5^{(1)} - \phi_5^{(0)}) + 2(\phi_6^{(1)} - \phi_6^{(0)}) - 9\phi_1^2]K_1^2 \\ K_7' &= \tfrac{1}{2}\psi_2^2[4(\phi_5^{(1)} - \phi_5^{(0)}) + 5(\phi_6^{(1)} - \phi_6^{(0)}) - 9\phi_1^2]K_1^2 \end{aligned} \quad (21)$$

where the quantities ψ and ϕ are analogous to those of § 3, and are given in Schofield (1980). First-order results for K_c and λ_T are readily obtained from (21), and appear in table 2. (Superscripts again indicate the order of the calculation.) In second order, a fixed point of (21) is found to be

$$(K_1^*, \ldots, K_7^*) = (1{\cdot}2435, 0{\cdot}1008, 0{\cdot}1743, -0{\cdot}0316, -0{\cdot}5129, 0{\cdot}1009, 0{\cdot}4358). \quad (22)$$

Table 2. Results of five-spin block calculation.

$\lambda_T^{(1)}$	$\lambda_T^{(2)}$	$\lambda_H^{(0)}$	$\lambda_H^{(1)}$	$K_c^{(1)}$	$K_c^{(2)}$
2·089	2·228	3·073	4·527	1·530	1·374
$\alpha^{(1)}$	$\alpha^{(2)}$	β	γ	$\delta^{(0)}$	$\delta^{(1)}$
−0·185	−0·010	0·124	1·761	2·306	15·175

Linearisation of the RG transformation about $(K_1^*, \ldots, K_7^*)$ yields a (7×7) matrix, the eigenvalues of which are found to be

$$\begin{array}{llll} \lambda_T = 2{\cdot}2275 & \lambda_2 = 0{\cdot}4039 & \lambda_3 = -0{\cdot}1267 + 0{\cdot}1444\mathrm{i} & \\ \lambda_4 = \lambda_3^* & \lambda_5 = 0{\cdot}1084 & \lambda_6 = -0{\cdot}0943 & \lambda_7 = 0{\cdot}0288. \end{array} \quad (23)$$

The quantity K_c is now readily determined by linear extrapolation, and is given in table 2.

For the field analysis, (17) again applies. In the zeroth-order analysis we obtain two eigenvalues $\lambda_H = 3{\cdot}0727$ and $\lambda_2 = -1{\cdot}3724$; we regard the former as the relevant eigenvalue of interest, the apparently relevant nature of the latter being regarded as an artifact of the approximation used. In the first-order analysis we find a single relevant eigenvalue λ_H (listed in table 2) and two irrelevant eigenvalues, $\lambda_2 = -0{\cdot}0726 + 0{\cdot}1160\mathrm{i}$ and $\lambda_3 = \lambda_2^*$.

The derived exponents appear in table 2, and the remarks of the last paragraph of § 3 again apply.

5. Summary

The results obtained for the critical exponents are seen, for both block constructions, to provide good evidence—relevant to the case of ferrimagnetism—supporting the usual form of the universality hypothesis. As one might expect, the fixe-spin block construction yields somewhat better evidence than the four-spin block case.

We have demonstrated the application of the RG method to a model system which is in some respects more complicated than those previously examined. However, the complexity of equations (16) and (21), together with the field recursion relations, shows the computational difficulties involved in such an exercise.

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